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ON THE OPTIMAL DISTRIBUTION OF THE RESISTIVITY TENSOR OF THE WORKING SUBSTANCE IN A MAGNETOHYDRODYNAMIC CHANNEL

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The analysis of the problem formulated and studied in [1, 2] is resumed. The conditions which the distribution of the resistivity of the working substance in a channel with finite electrodes must satisfy in order for the current in the external circuit to reach its maximum are investigated. The resistivity is assumed to be a tensor function of the coordinates; the tensor is assumed to be symmetric and its principal values to be piecewise-continuously differentiable functions.

1. Formulation of the problem. We consider a flat channel (Fig. 1) of width 2δ whose walls are dielectric everywhere except for two segments of equal length 2λ facing each other at opposite sides of the channel; these segments are made of an ideally conductive material. The conductive segments are connected through the load R.

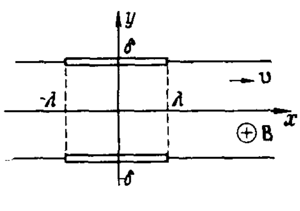

Fig. 1

The working substance characterized by the resistivity tensor $\mathrm{P}_0(x, y)$ which varies from point to point is moving in the channel at the velocity $\mathbf{v}\,(V(y),\,0,\,0)$. We assume that this center is symmetric; let $\rho_1(x, y)$, $\rho_2(x, y)$ be its principal values and α, β the corresponding principal axes. Denoting the angle between the positive direction of the x-axis and the α-axis (*) by $\gamma(x, y)$, we can find the Cartesian components of the tensor P_0 from the formulas

$$\rho_{xx} = {}^1/_2\,[\rho_1 + \rho_2 + (\rho_1 - \rho_2)\cos 2\gamma], \quad \rho_{yy} = {}^1/_2\,[\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\gamma] \tag{1.1}$$

$$\rho_{xy} = \rho_{yx} = {}^1/_2\,(\rho_1 - \rho_2)\sin 2\gamma$$

Imposition of a magnetic field $\mathbf{B}\,(0,\,0, -B\,(x))$ causes an electric current of density $\mathbf{j}$ to flow in the channel (the Cartesian coordinates of this vector will be denoted

*) We assume that the α- and β-axes form a right-handed system.

by ζ^1, ζ^2) and a current I given by the formula

$$I = \int_{-\lambda}^{\lambda} \zeta^2 (x, \pm \delta)\, dx \tag{1.2}$$

to flow in the external circuit.

The equations describing the current distribution in the channel can be written as [1, 2]

$$\operatorname{div} \mathbf{j} = 0, \qquad \mathrm{P}_0 \cdot \mathbf{j} = -\operatorname{grad} z^1 + \frac{1}{c} \mathbf{v} \times \mathbf{B} \tag{1.3}$$

Here z^1 represents the electric field potential.

Introducing the current function $z^2(x, y)$ by way of the relation $\mathbf{j} = -\operatorname{rot} \mathbf{i}_3 z^2$, we can rewrite Eq. (1.3) in the following standard form (where z_x^1, z_y^1, etc., denote derivatives):

$$\begin{aligned} z_x^1 &= -\rho_{xx}\zeta^1 - \rho_{xy}\zeta^2 & z_x^2 &= \zeta^2 \\ z_y^1 &= -\rho_{yx}\zeta^1 - \rho_{yy}\zeta^2 + c^{-1} VB, & z_y^2 &= -\zeta^1 \end{aligned} \tag{1.4}$$

Relation (1.2) becomes

$$I = z^2(\lambda, \pm \delta) - z^2(-\lambda, \pm \delta) \tag{1.5}$$

To Eqs. (1.4) we add boundary conditions expressing the properties of the channel walls, the conditions at infinity, and Ohm's law for the external circuit; these conditions are of the form

$$z^1(x, \pm \delta) = z_{\pm}^1 = \text{const} \qquad \text{(at the electrodes)} \tag{1.6}$$

$$z^2(x, \pm\delta)|_{x>\lambda} = z_+^2 = \text{const}, \quad z^2(x, \pm\delta)|_{x<-\lambda} = z_-^2 = \text{const} \;\; \text{(at the insulators)}$$

$$z_x^2(\pm\infty, y) = z_y^2(\pm\infty, y) = 0, \qquad z_+^1 - z_-^1 = R(z_+^2 - z_-^2)$$

We are to determine the pair of piecewise-continuously differentiable functions $\rho_1(x, y), \rho_2(x, y)$ subject to inequalities

$$0 < \rho_{\min} \leqslant \rho_i(x, y) \leqslant \rho_{\max} \leqslant \infty, \qquad i = 1, 2 \tag{1.7}$$

and the piecewise-continuously differentiable function $\gamma(x, y)$ which maximize functional (1.5).

The above problem differs from those considered in [1, 2] by the tensor character of the resistivity of the working substance. As we shall see below, this fact is of considerable importance (the optimal distribution in practically interesting cases exists in the class of tensor functions but not in the class of scalar functions).

2. The necessary steadystate conditions. Let us introduce the Lagrange multipliers $\xi_1, \eta_1, \xi_2, \eta_2$ corresponding to the four equations (1.4) and construct the function H after expressing the Cartesian components of the tensor P_0 in terms of the controlling functions ρ_1, ρ_2, γ according to formulas (1.1). We have

$$\begin{aligned} H = &-{}^1\!/_2\, \xi_1 \{[\rho_1 + \rho_2 + (\rho_1 - \rho_2)\cos 2\gamma]\zeta^1 + (\rho_1 - \rho_2)\sin 2\gamma\, \zeta^2\} - \\ &- {}^1\!/_2\, \eta_1 \{(\rho_1 - \rho_2)\sin 2\gamma\, \zeta^1 + [\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\gamma]\zeta^2 - c^{-1} VB\} + \\ &+ \xi_2 \zeta^2 - \eta_2 \zeta^1 - \sum_{i=1}^{2} \mu_i [(\rho_{\max} - \rho_i)(\rho_i - \rho_{\min}) - \rho_{i*}^2] \end{aligned} \tag{2.1}$$

The multipliers μ_i, $i = 1, 2$ correspond to restrictions (1.7) written in the form of equivalent equations [3].

The steadystate conditions are given by the formulas [3]

$$\xi_{1x} + \eta_{1y} = 0, \qquad \xi_{2x} + \eta_{2y} = 0 \tag{2.2}$$

$$\partial H / \partial \zeta_1 = 0, \ \partial H / \partial \zeta^2 = 0 \tag{2.3}$$

$$\partial H / \partial \rho_1 = 0, \quad \partial H / \partial \rho_2 = 0, \ \partial H / \partial \gamma = 0$$

$$\partial H / \partial \rho_{1*} = 0, \qquad \partial H / \partial \rho_{2*} = 0 \tag{2.4}$$

Let us introduce the functions $\omega_1(x, y)$, $\omega_2(x, y)$ by way of the relations

$$\xi_1 = -\omega_{1y}, \quad \xi_2 = -\omega_{2y}; \qquad \eta_1 = \omega_{1x}, \quad \eta_2 = \omega_{2x} \tag{2.5}$$

Equations (2.2) are satisfied identically in this case; as regards Eqs. (2.3) and (2.4), they can be written as (we omit the obvious intervening operations)

$$\rho_2 \omega_{1\alpha} + \omega_{2\beta} = 0 \tag{2.6}$$

$$\rho_1 \omega_{1\beta} - \omega_{2\alpha} = 0 \tag{2.7}$$

$$\begin{gathered} \rho_1^{-1} j_\alpha \omega_{2\alpha} + \mu_1 (2\rho_1 - \rho_{\max} - \rho_{\min}) = 0 \\ \rho_2^{-1} j_\beta \omega_{2\beta} + \mu_2 (2\rho_2 - \rho_{\max} - \rho_{\min}) = 0 \end{gathered} \tag{2.8}$$

$$(\rho_1 - \rho_2)(\rho_1 \omega_{2\beta} j_\alpha + \rho_2 \omega_{2\alpha} j_\beta) = 0 \tag{2.9}$$

$$\mu_1 \rho_{1*} = 0, \qquad \mu_2 \rho_{2*} = 0 \tag{2.10}$$

Here

$$\omega_{i\alpha} = \omega_{ix} \cos \gamma + \omega_{iy} \sin \gamma, \qquad \omega_{i\beta} = -\omega_{ix} \sin \gamma + \omega_{iy} \cos \gamma \qquad (i = 1, 2) \tag{2.11}$$

$$j_\alpha = \zeta^1 \cos \gamma + \zeta^2 \sin \gamma, \ \ j_\beta = -\zeta^1 \sin \gamma + \zeta^2 \cos \gamma \tag{2.12}$$

The functions $\omega_{i\alpha}$, $\omega_{i\beta}$ and j_α, j_β are the physical components of the vectors grad ω_i and $\mathbf{j}$ along the α- and β-axes.

The boundary conditions for the functions $\omega_i(x, y)$ $(i = 1, 2)$ must be constructed with allowance for relations (1.5) and (1.6) (see [1]). We have

$$\omega_2(x, \pm\delta) = \omega_{2\pm} = \text{const} \quad \text{at the electrodes}$$

$$\omega_1(x, \pm\delta)|_{x>\lambda} = \omega_{1+} = \text{const}, \ \omega_1(x, \pm\delta)|_{x<-\lambda} = \omega_{1-} = \text{const} \ \ \text{at the insulators}$$

$$\operatorname{grad} \omega_1 = \operatorname{grad} \omega_2 = 0 \quad \text{at infinity}$$

$$\omega_{2+} - \omega_{2-} + 1 = R(\omega_{1+} - \omega_{1-}) \tag{2.13}$$

Relations (2.6), (2.7), (2.13) for the functions ω_1, ω_2 can be interpreted as the equations and boundary conditions describing the distribution in the channel of fictitious currents of density $-\mathrm{P}_0^{-1} \cdot \operatorname{grad} \omega_2$ due to the "potential difference" $\omega_{2+} - \omega_{2-} + 1$ at the electrodes in the absence of other electromotive forces. The function ω_1 in this case plays the role of the corresponding "current function".

3. The necessary Weierstrass condition. The simplest way to construct this condition is to make use of the following expression for the increment δI of the functional I associated with the arbitrary variation $\delta \mathrm{P}$ of the tensor P_0:

$$\delta I = -\iint_S [\xi_1 (\delta\rho_{xx} Z^1 + \delta\rho_{xy} Z^2) + \eta_1 (\delta\rho_{yx} Z^1 + \delta\rho_{yy} Z^2)] \, dx \, dy \tag{3.1}$$

Here Z^1, Z^2 are the Cartesian components of the current density vector $\mathbf{J}$ corresponding to the permissible resistivity tensor $\mathrm{P} = \mathrm{P}_0 + \delta\mathrm{P}$ (from now on we shall use capital letters to denote permissible quantities and the corresponding small letters to denote optimal quantities; the sole exceptions will be the optimal resistivity tensor denoted by P_0 and the permissible resistivity tensor denoted by P.).

Formula (3.1) is exact; it is readily derivable from the initial equations with allowance for the steadystate conditions.

Making use of formulas (2.5)–(2.7), we can transform the integrand in (3.1) into

$$-(P_0^{-1}\cdot \operatorname{grad}\omega_2)(\delta P\cdot \mathbf{J})$$

Introducing the dyad

$$\mathbf{J}\,(P_0^{-1}\cdot \operatorname{grad}\omega_2)$$

we obtain the following formula for the increment δI of the functional:

$$\delta I = \iint_S Sp\,(\delta P\cdot \mathbf{J}\,(P_0^{-1}\cdot \operatorname{grad}\omega_2))\,dx\,dy \tag{3.2}$$

The functional I attains its maximum on the control P_0 if and only if the quantity δI is made nonpositive for all permissible δP and $\mathbf{J}$.

The vector $\mathbf{J}$ occurring in expression (3.2) can be obtained by solving the initial boundary value problem in which the tensor P_0 has been replaced by the permissible tensor P. The difficulties involved in solving this problem are not less formidable than those of solving the initial one; the condition $\delta I \leqslant 0$ turns out to be ineffective with such a general method of variation. In order to obtain an effective condition we must specialize the variation of the tensor δP. Specifically, let us assume that this variation differs from zero only within a narrow strip of width b and length l, ($\varepsilon = b / l$ is the small parameter of the problem). The permissible tensor $P = P_0 + \delta P$ must satisfy restrictions (1.7); it is otherwise arbitrary.

Such a special variation clearly requires us to compute the vector $\mathbf{J}$ within the strip only; moreover, it is sufficient to find the principal part of this vector which is linear in ε. If the strip is sufficiently narrow and if the field of the vector $\mathbf{j}$ inside the strip is free of singularities, then the principal linear part of the vector $\mathbf{J}$ can be computed by assuming that the strip lies in the external homogeneous current field $\mathbf{j}$. Clearly, the increment δP can then be regarded as a constant tensor within the strip.

The solution of the latter problem is well known. Let us assume that the strip is oriented along the axis $\mathbf{t}$, that it has the exterior normal $\mathbf{n}$ (Fig. 2), and that its resistivity is characterized by the tensor $P = P_0 + \delta P$. If the strip is situated in the homogeneous field $\mathbf{j}$ of currents flowing in a medium with the resistivity P_0, then the components of the current density inside the strip can be determined from the formulas

$$J_n = j_n + O(\varepsilon) \tag{3.3}$$

$$J_t = j_t + \frac{\rho_{tt} - P_{tt}}{P_{tt}}\, j_t + \frac{\rho_{tn} - P_{tn}}{P_{tt}}\, j_n + O(\varepsilon)$$

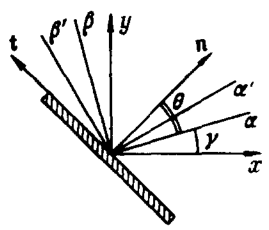

Fig. 2

We shall use these formulas to eliminate the vector $\mathbf{J}$ from the left side of the inequality

$$\mathrm{Sp}\,(\delta P\cdot \mathbf{J}\,(P_0^{-1}\cdot \operatorname{grad}\omega_2)) \leqslant 0 \tag{3.4}$$

which now follows from (3.2) and is fulfilled almost everywhere as the necessary condition of a strong relative maximum.

Converting to the principal axes α and β of the tensor P_0 we can rewrite this inequality in the following equivalent form (where E is the Weierstrass function):

$$E = -[(P_{\alpha\alpha} - \rho_1) J_\alpha + P_{\alpha\beta} J_\beta] \rho_1^{-1} \omega_{2\alpha} - [P_{\beta\alpha} J_\alpha + (P_{\beta\beta} - \rho_2) J_\beta] \rho_2^{-1} \omega_{2\beta} \geqslant 0 \tag{3.5}$$

The components J_α, J_β of the vector $\mathbf{J}$ must be expressed in terms of j_α, j_β, and the components $P_{\alpha\alpha}$, $P_{\alpha\beta}$, $P_{\beta\beta}$ of the tensor P in terms of the quantities characterizing this tensor along its own principal axes. As a result, the function E depends on the principal values P_1, P_2 of the tensor P, on the angle $\lambda = \delta\gamma$ between the principal axes A and α of the tensors P and P_0, and also on the angle θ between the principal direction $\mathbf{n}$ of the normal to the strip and the axis α (Fig. 2). The remaining quantities occurring in the expression for E characterize the optimal mode and must be assumed to be fixed.

Carrying out the necessary computations (see Appendix), we arrive at the following equivalent expression for inequality (3.5):

$$M \equiv \frac{P_1 - \rho_1}{\rho_1} K_\alpha \omega_{2\alpha} + \frac{P_2 + \rho_2}{\rho_2} K_\beta \omega_{2\beta} - (P_1 - P_2)(j_\alpha \omega_{2\alpha} - j_\beta \omega_{2\beta})(1 - \cos 2\lambda) +$$
$$+ (P_1 - P_2) \frac{\rho_1 + \rho_2}{2\rho_1 \rho_2} (\rho_1 j_\alpha \omega_{2\beta} + \rho_2 j_\beta \omega_{2\alpha}) \sin 2\lambda \leqslant 0 \tag{3.6}$$

Here

$$K_\alpha = j_\alpha [P_2 + \rho_1 + (P_2 - \rho_1) \cos 2\theta] + j_\beta (P_2 - \rho_2) \sin 2\theta \tag{3.7}$$
$$K_\beta = j_\beta [P_1 + \rho_2 - (P_1 - \rho_2) \cos 2\theta] + j_\alpha (P_1 - \rho_1) \sin 2\theta$$

Let us investigate inequality (3.6). In accordance with conditions (2.10) we must discriminate between

1) nonsingular modes: $\mu_1 \neq 0$, $\mu_2 \neq 0$; $\rho_{1*} = \rho_{2*} = 0$;

2) modes singular in one of the controlling functions ρ_1, ρ_2: $\mu_1 \neq 0$, $\mu_2 = 0$ (so that $\rho_{1*} = 0$) or $\mu_2 \neq 0$, $\mu_1 = 0$ (so that $\rho_{2*} = 0$);

3) modes singular in both controlling functions ρ_1, ρ_2 : here $\mu_1 = \mu_2 = 0$.

The latter class of cases must be excluded forthwith, since Eqs. (2.6)–(2.8) imply in this instance that $\omega_1 = \text{const}$, $\omega_2 = \text{const}$, which is a variant devoid of interest.

4. Nonsingular modes. The controlling functions ρ_1, ρ_2 can assume the limiting values $\rho_{\max}$ or $\rho_{\min}$ only. Let us suppose that the control mode is essentially anisotropic, i.e. that $\rho_1 \neq \rho_2$.

Now, setting first $\lambda = 0$, $P_1 = \rho_1$, $P_2 \neq \rho_2$ and then $\lambda = 0$, $P_2 = \rho_2$, $P_1 \neq \rho_1$ in (3.6) and recalling (2.9), we arrive at inequalities characterizing the possible modes:

$$\rho_1 = \rho_{\max}, \quad \rho_2 = \rho_{\min}, \qquad \text{if} \quad j_\alpha \omega_{2\alpha} \geqslant 0, \quad j_\beta \omega_{2\beta} \leqslant 0 \tag{4.1}$$

$$\rho_1 = \rho_{\min}, \quad \rho_2 = \rho_{\max}, \qquad \text{if} \quad j_\alpha \omega_{2\alpha} \leqslant 0, \quad j_\beta \omega_{2\beta} \geqslant 0 \tag{4.2}$$

Conditions (4.2) can be obtained from (4.1) by interchanging the roles of the axes α and β.

Inequalities (4.1), (4.2) are necessary to the fulfillment of inequality (3.6); let us show that they are also sufficient. Considering conditions (4.1), we compute the left side of inequality (3.6) $\rho_1 = \rho_{\max}$, $\rho_2 = \rho_{\min}$. Making use of formulas (3.7) and (2.9), we obtain

$$M = \frac{P_1 - \rho_{\max}}{\rho_{\max}} j_\alpha \omega_{2\alpha} [P_2 + \rho_{\max} + (P_2 - \rho_{\max}) \cos 2\theta] +$$

$$+\frac{P_2-\rho_{\min}}{\rho_{\min}} j_\beta\, \omega_{2\beta} [P_1 + \rho_{\min} - (P_1 - \rho_{\min}) \cos 2\theta] - \qquad \text{(cont.)}$$
$$-(P_1 - P_2)(j_\alpha\, \omega_{2\alpha} - j_\beta\, \omega_{2\beta})(1-\cos 2\lambda)$$

The inequality $M \leqslant 0$ must be fulfilled for all permissible P_1, P_2, θ and λ. We have

$$\max_\lambda M = \begin{cases} M(\lambda=0), & \text{if} \quad P_1 \geqslant P_2 \\ M(\lambda=0) - 2(P_1 - P_2)(j_\alpha\, \omega_{2\alpha} - j_\beta\, \omega_{2\beta}), & \text{if} \quad P_1 \leqslant P_2 \end{cases}$$

It is easy to show by direct computation that the inequality $\max_\lambda M \leqslant 0$ is fulfilled in both of the above cases whatever the permissible values of P_1, P_2 and θ.

A similar conclusion can be drawn in respect of mode (4.2).

The above analysis shows, among other things, that in this case the orientation of the strip of variation has no effect on the form of the minimum conditions. This fact (which also obtains in the other cases analyzed below) is due to the anisotropic character of the control variation. Inside the strip the optimal tensor P_0 is replaced by the permissible tensor P with new principal directions which can be arbitrary; thereafter, rotation of the strip itself does not alter the result in any way. This statement does not apply in the case of the problem with a scalar control (see [1, 2]), where the optimal scalar function ρ is replaced by the scalar control P in the strip of variation; moreover, the slope of the strip occurs explicitly in the formula for the increment of the functional (as the sole factor characterizing the variation anisotropy). The way in which this affects the final form of the mimimum conditions in the problem with a scalar control is discussed in [1].

Returning to the matter of nonsingular anisotropic modes, we conclude that they can be characterized by the relative disposition of the vectors $\mathbf{j}$, grad ω_2 and the α- and β-axes (see Figs. 3 and 4). The vectors $\mathbf{j}$ and grad ω_2 lie in the neighboring quadrants separated either by the α-axis (mode (4.1)) or by the β-axis (mode (4.2)) (by the axis corresponding to the principal value $\rho_{\max}$ of the tensor P_0 in both cases).

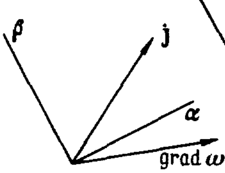

Fig. 3

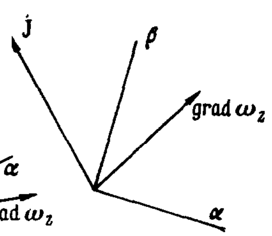

Fig. 4

Let us consider the possibility of nonsingular isotropic modes. To be specific, let $\rho_1 = \rho_2 = \rho_{\max}$. Then, clearly,

$$j_\alpha\, \omega_{2\alpha} \geqslant 0, \qquad j_\beta\, \omega_{2\beta} \geqslant 0 \qquad (4.3)$$

These inequalities are readily obtainable from (3.6) by setting $\lambda = 0$ and then $P_2 = \rho_2$, $P_1 \neq \rho_1$ and $P_1 = \rho_1$, $P_2 \neq \rho_2$ in that order.

On the other hand, Eq. (2.9) is satisfied identically for $\rho_1 = \rho_2 = \rho_{\max}$, since any pair of mutually perpendicular directions can be regarded as the principal directions in the isotropic mode. This implies that inequalities (4.3) must be fulfilled for any pair of mutually perpendicular directions α and β. This is possible only if the vectors $\mathbf{j}$ and grad ω_2 are related by the equation

$$\mathbf{j} = F \operatorname{grad} \omega_2 \qquad (4.4)$$

where $F = F(x, y)$ is a nonnegative function.

It is not difficult to verify that the inequality $M \leqslant 0$ is fulfilled in this case.

The case $\rho_1 = \rho_2 = \rho_{\min}$ can be disposed of in similar fashion; the function F is nonpositive in this instance.

It is interesting to compare condition (4.4) with the condition of realization of the control $\rho = \rho_{max}$ in the problem with a scalar control. In [1] we showed that this condition can be written as

$$\chi \leqslant \arccos p, \qquad p = (\rho_{max} - \rho_{min}) / (\rho_{max} + \rho_{min}) \tag{4.5}$$

where χ is the angle between the vectors $\mathbf{j}$ and $\mathrm{grad}\ \omega_2$. Comparison shows that (4.4) is equivalent to the requirement that $\chi = 0$. If this requirement is fulfilled, then inequality (4.5) is satisfied; the converse statement is not valid. This conclusion is quite natural: considering anisotropic variations in the neighborhood of an isotropic control, we necessarily arrive at a stricter minimum condition.

5. Singular modes. Let us suppose that $\mu_1 \neq 0$, $\mu_2 = 0$. Equations (2.8) imply that $j_\beta \omega_{2\beta} = 0$. Let $\omega_{2\beta} = 0$; eliminating the trivial case $\omega_{2\alpha} = 0$, we infer from Eq. (2.9) that $j_\beta = 0$. Conversely, if $j_\beta = 0$, then, eliminating the trivial case $j_\alpha = 0$, we infer from (2.9) that $\omega_{2\beta} = 0$.

Thus, the special case $\omega_{2\beta} = 0$ is characterized by the fact that at the same time $j_\beta = 0$, i.e. that the vectors $\mathrm{grad}\ \omega_2$ and $\mathbf{j}$ have the same direction as the principal axis α of the tensor P_0. The sign of the scalar product $\mathbf{j} \cdot \mathrm{grad}\ \omega_2$ is easy to determine by means of inequality (3.6); we obtain

$$\rho_1 = \rho_{max}, \text{ if } \quad j_\alpha \omega_{2\alpha} \geqslant 0 \quad \text{(the vectors } \mathbf{j} \text{ and } \mathrm{grad}\ \omega_2 \text{ are parallel)}$$

$$\rho_1 = \rho_{min}, \text{ if } \quad j_\alpha \omega_{2\alpha} \leqslant 0 \quad \text{(the vectors } \mathbf{j} \text{ and } \mathrm{grad}\ \omega_2 \text{ are antiparallel)}$$

The control ρ_2 is arbitrary in the singular mode under consideration. This fact is general in character and is a consequence of the fact that if the vector $\mathbf{j}$ is directed along one of the principal axes of the tensor P_0, then Eqs. (1.4) do not depend on the principal value corresponding to the other principal axis (to prove this we need merely write out these equations in the principal axes of the tensor P_0; see formulas (A.10) in the Appendix). The singular mode $\mu_1 = 0$, $\mu_2 \neq 0$ can be disposed of in similar fashion.

The reults of the last three sections can be summarized as follows.

Theorem. The optimal controls in the problem of Sect. 1 are characterized by the following possible modes.

1. Nonsingular controls

1a. Anisotropic

$$\rho_1 = \rho_{max}, \quad \rho_2 = \rho_{min}, \quad \text{if} \quad j_\alpha \omega_{2\alpha} \geqslant 0, \ j_\beta \omega_{2\beta} \leqslant 0$$
$$\rho_1 = \rho_{min}, \quad \rho_2 = \rho_{max}, \quad \text{if} \quad j_\alpha \omega_{2\alpha} \leqslant 0, \ j_\beta \omega_{2\beta} \geqslant 0$$

1b. Isotropic

$$\rho_1 = \rho_2 = \rho_{max}, \quad \text{if} \quad \mathbf{j} = F \,\mathrm{grad}\, \omega_2, \ F \geqslant 0$$
$$\rho_1 = \rho_2 = \rho_{min}, \quad \text{if} \quad \mathbf{j} = F \,\mathrm{grad}\, \omega_2, \ F \leqslant 0$$

2. Singular controls

2a. $\omega_{2\beta} = 0, \ j_\beta = 0$

$$\rho_1 = \begin{cases} \rho_{max}, & \text{if} \quad j_\alpha \omega_{2\alpha} \geqslant 0, \\ \rho_{min}, & \text{if} \quad j_\alpha \omega_{2\alpha} \leqslant 0, \end{cases} \qquad \rho_2 \text{ is arbitrary}$$

2b. $\omega_{2\alpha} = 0,\ j_\alpha = 0$

$$\rho_1 \text{ is arbitrary}; \qquad \rho_2 = \begin{cases} \rho_{\max}, & \text{if} \quad j_\beta \omega_{2\beta} \geqslant 0 \\ \rho_{\min}, & \text{if} \quad j_\beta \omega_{2\beta} \leqslant 0 \end{cases}$$

In both of the latter cases we can assume, in particular, that $\rho_2 = \rho_1$. This results in an isotropic control mode.

6. The Weierstrass-Erdmann condition and the Weierstrass condition. Let us consider the Weierstrass-Erdmann condition, which is fulfilled on the line separating domains with different control modes in the variables ρ_1, ρ_2, γ. Assuming that this line is sufficiently smooth, we can write the Weierstrass-Erdmann condition in the form [3]

$$\omega_{1t}\,[z_n{}^1]_-^+ + \omega_{2t}\,[z_n{}^2]_-^+ = 0 \tag{6.1}$$

The symbol $[\ \ ]_-^+$ denotes the difference between the extreme values of the corresponding quantity to the left and right of the line of separation; $\mathbf{t}$ and $\mathbf{n}$ are the unit vectors of the tangent and the normal to this line. The derivatives ω_{1t} and ω_{2t} are continuous at the line of separation. It does not matter therefore on which side of the line the extreme values of these derivatives are taken.

Let us transform Eqs. (6.1), converting to the local Cartesian coordinates α, β (different on each side of the line) by means of formulas of the type (2.11) and (2.12). We assume that the function $\boldsymbol{VB}$ is continuous. Taking account of the continuity of the derivatives ω_{1t}, ω_{2t} and making use of Eqs. (2.6), (2.7), we can rewrite condition (6.1) in the form

$$\begin{aligned} &[\omega_{2\alpha} j_\alpha \cos^2\theta + \omega_{2\beta} j_\beta \sin^2\theta + (\rho_1/\rho_2)\,\omega_{2\beta} j_\alpha \sin\theta\cos\theta + \\ &+ (\rho_2/\rho_1)\,\omega_{2\alpha} j_\beta \sin\theta\cos\theta]_-^+ - [\omega_{2\alpha} j_\alpha \sin^2\theta + \omega_{2\beta} j_\beta \cos^2\theta - \\ &- \omega_{2\beta} j_\alpha \sin\theta\cos\theta - \omega_{2\alpha} j_\beta \sin\theta\cos\theta]_-^+ = 0 \end{aligned} \tag{6.2}$$

In carrying out further transformations of this equation we must bear in mind the possible differences between the control modes on the two sides of the jump line. Let us consider several of the possible cases.

1a), (1a): both modes are nonsingular and anisotropic; here

$$\rho_{1+} = \rho_{\max}, \quad \rho_{2+} = \rho_{\min}; \qquad \rho_{1-} = \rho_{\min}, \quad \rho_{2-} = \rho_{\max} \tag{6.3}$$

Making use of steadystate condition (2.9) for the left- and right-hand extreme values, we can reduce (6.2) to

$$[\cos 2\theta\,(j_\alpha\omega_{2\alpha} - j_\beta\omega_{2\beta})]_-^+ = 0 \tag{6.4}$$

1a), (1.b): anisotropic and isotropic modes; here

$$\rho_{1+} = \rho_{\max}, \quad \rho_{2+} = \rho_{\min}; \qquad \rho_{1-} = \rho_{\max}, \quad \rho_{2-} = \rho_{\max} \tag{6.5}$$

Steadystate condition (2.9) is effective for an anisotropic state only. It is satisfied identically in the case of an anisotropic mode. We obtain

$$[\cos 2\theta\,(j_\alpha\omega_{2\alpha} - j_\beta\omega_{2\beta})]_-^+ - \sin 2\theta_-\,(j_\alpha\omega_{2\beta} + j_\beta\omega_{2\alpha})_- = 0 \tag{6.6}$$

This result does not depend on which extreme values ($\rho_{\max}$ or $\rho_{\min}$) the functions ρ_1, ρ_2 assume to each side of the jump line (within the limits of combination of control modes (1a) and (1b), of course).

1b), (1b): both modes isotropic; here

$$\rho_{1+} = \rho_{\max}, \quad \rho_{2+} = \rho_{\max}; \qquad \rho_{1-} = \rho_{\min}, \quad \rho_{2-} = \rho_{\min} \tag{6.7}$$

Condition (6.1) can be written as

$$[\cos 2\theta\,(j_\alpha\omega_{2\alpha} - j_\beta\omega_{2\beta})]_-^+ + [\sin 2\theta\,(j_\alpha\omega_{2\beta} + j_\beta\omega_{2\alpha})]_-^+ = 0 \tag{6.8}$$

similar formulas characterize the lines separating singular and nonsingular modes, etc.

Relations (6.4), (6.6), (6.8), etc. must be made compatible with steadystate conditions (2.9) and Weierstrass inequalities (see the Theorem above) which the extreme values of the vectors $\mathbf{j}$ and $\mathbf{grad}\ \omega_2$ satisfy. The requirement that these relations be compatible isolates certain combinations of extreme values which satisfy all of the necessary minimum conditions.

We can obtain the necessary expressions by making use of formulas (A.11) of the Appendix which relate the extreme values of the components of the vectors $\mathbf{j}_+$ and $\mathbf{j}_-$ and also of the vectors $(\mathbf{grad}\ \omega_2)_+$ and $(\mathbf{grad}\ \omega_2)_-$.

Let us consider the transition (1a), (1a) defined by (6.3). Making use of the aforementioned formulas, we construct Eq. (2.9) for the minus-extreme values; after transformations carried out with due allowance for condition (2.9) for the plus-extreme values, we obtain the equation

$$\rho_{\min}\rho_{\max} a \cos(\theta_+ - \theta_-)(j_\alpha\omega_{2\alpha} - j_\beta\omega_{2\beta})_+ = 0 \tag{6.9}$$

$$a = \rho_{\min}\sin\theta_-\cos\theta_+ - \rho_{\max}\cos\theta_-\sin\theta_+$$

Equation (6.9) shows that two cases are possible: $a = 0$ and $(j_\alpha\omega_{2\alpha} - j_\beta\omega_{2\beta})_+ = 0$ (the case $\cos(\theta_+ - \theta_-) = 0$ is trivial, as it corresponds to the absence of a jump). On the other hand, the Theorem implies fulfillment of the inequalities

$$(j_\alpha\omega_{2\alpha})_+ \geqslant 0, \quad (j_\beta\omega_{2\beta})_+ \leqslant 0; \qquad (j_\alpha\omega_{2\alpha})_- \leqslant 0, \quad (j_\beta\omega_{2\beta})_- \geqslant 0 \tag{6.10}$$

We can use formulas (A.11) of the Appendix and conditions (2.9) for the plus-extreme values to transform the second pair of the above inequalities into

$$(j_\alpha\omega_{2\alpha})_+\rho_{\max}\rho_{\min}\cos^2(\theta_+ - \theta_-) + a^2(j_\beta\omega_{2\beta})_+ \leqslant 0 \tag{6.11}$$

$$(j_\beta\omega_{2\beta})_+\rho_{\max}\rho_{\min}\cos^2(\theta_+ - \theta_-) + a^2(j_\alpha\omega_{2\alpha})_+ \geqslant 0$$

If $a = 0$, then these inequalities are compatible with the first pair of inequalities (6.10) only in the case $(j_\alpha\omega_{2\alpha})_+ = (j_\beta\omega_{2\beta})_+ = 0$. The same statement is valid when $(j_\alpha\omega_{2\alpha})_+ - (j_\beta\omega_{2\beta})_+ = 0$. Recalling formula (2.9) for the plus-extreme values, we conclude that the following conditions are fulfilled on the line separating nonsingular essentially anisotropic modes:

$$\text{either } \mathbf{j}_+ = \mathbf{j}_- = 0, \qquad \text{or } (\mathbf{grad}\ \omega_2)_+ = (\mathbf{grad}\ \omega_2) = 0$$

Let us consider the transition (1a), (1b) in accordance with (6.5). We must have (see Theorem)

$$(j_\alpha\omega_{2\alpha})_+ \geqslant 0, \quad (j_\beta\omega_{2\beta})_+ \leqslant 0, \quad (j_\alpha)_- = F(\omega_{2\alpha})_-, \quad (j_\beta)_- = F(\omega_{2\beta})_-, \quad F \geqslant 0 \tag{6.12}$$

If $F > 0$, then, proceeding as in our derivation of relation (6.9), we obtain the equation

$$\rho_{\max}(j_\alpha\omega_{2\beta})_+ - \rho_{\min}(j_\beta\omega_{2\alpha})_+ = 0$$

Comparing this with condition (2.9) for the plus-extreme values, we obtain

$$(j_\alpha\omega_{2\beta})_+ = (j_\beta\omega_{2\alpha})_+ = 0$$

This implies that either $(j_\beta)_+ = 0$, $(\omega_{2\beta})_+ = 0$, or $(j_\alpha)_+ = 0$, $(\omega_{2\alpha})_+ = 0$. The second possibility together with the condition $(j_\beta\omega_{2\beta})_+ \leqslant 0$ yields the inequality $(j_\beta\ \omega_{2\beta})_- < 0$ (this is easy to show with the aid of formulas (A.11) of the Appendix). The latter inequality contradicts the condition $F \geqslant 0$ (provided we leave aside the trivial possibility of simultaneous fulfillment of the conditions $(j_\beta)_+ = (\omega_{2\beta})_+ = 0$).

It remains for us to investigate the case $(j_\beta)_+ = (\omega_{2\beta})_+ = 0$. It is easy to show that relations (6.12) are fulfilled in this case. By analogy with the foregoing analysis, we find that

$$(j_\alpha\omega_{2\alpha})_- > 0, \quad (j_\beta)_- = (\omega_{2\beta})_- = 0$$

We see therefore that the vectors $\mathbf{j}$ and grad ω_2 experience no change either in magnitude or in direction in passing from the isotropic control mode domain into that of a nonsingular essentially anisotropic mode. The reason for this lies in the fact that the direction of the aforementioned vectors at the line of separation on the anisotropic mode is the same as that of the principal axis along which the resistivity remains the same as in the isotropic mode zone (in this case ρ_{max}).

Let us consider the transition (1b), (1b) in accordance with (6.7). Applying formulas (A.11) of the Appendix and the Teorem to this case, we find that at the jump line we must have either

$$\mathbf{j}_+ = \mathbf{j}_- = 0$$

or

$$(\operatorname{grad} \omega_2)_+ = (\operatorname{grad} \omega_2)_- = 0$$

These conditions are the same as the corresponding conditions at the line separating two nonsingular anisotropic modes (*).

The conditions at the lines separating nonsingular and singular modes, etc., can be obtained in similar fashion.

7. The asymptotic case $\rho_{max} = \infty$. Here and in subsequent sections we shall consider the case where the upper limit ρ_{max} in inequalities (1.7) is equal to infinity. We propose to construct asymptotic equations describing the optimal mode and to describe some of the principal properties of the optimal control.

Let us consider steadystate condition (2.9) with reference to the control γ,

$$\rho_1 j_\alpha\omega_{2\beta} + \rho_2 j_\beta\omega_{2\alpha} = 0$$

Converting from the principal axes α, β to the Cartesian axes x, y, we can rewrite this equation as

$$(\rho_1\zeta^2\omega_{2x} + \rho_2\zeta^1\omega_{2y})\operatorname{tg}^2\gamma - (\rho_1 + \rho_2)(\omega_{2y}\zeta^2 - \omega_{2x}\zeta^1)\operatorname{tg}\gamma - \\ - (\rho_1\omega_{2y}\zeta^1 + \rho_2\omega_{2x}\zeta^2) = 0 \qquad (7.1)$$

The Theorem formulated and proved in Sect. 5 enables us to set down a rule for choosing the roots of this equation. If the limiting values ρ_{max}, ρ_{min} are both finite, then direct computation (see Appendix) shows that the root $(\operatorname{tg}\gamma)_1$ (the upper sign in the formula for the roots) corresponds to the inequalities $j_\alpha\omega_{2\alpha} \geqslant 0$, $j_\beta\omega_{2\beta} \leqslant 0$, and the root $(\operatorname{tg}\gamma)_2$ (the lower sign in the formula for the roots) to the inequalities $j_\alpha\omega_{2\alpha} \leqslant 0$, $j_\beta\omega_{2\beta} \geqslant 0$. According to the Theorem we must take $\rho_1 = \rho_{max}, \rho_2 = \rho_{min}$ in the first case and $\rho_1 = \rho_{min}$, $\rho_2 = \rho_{max}$ in the second.

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*) The need to make the Weierstrass-Erdmann condition compatible with the Weierstrass condition on both sides of the jump line also arises in the case of a scalar control [1, 2]. This requirement implies the following condition: the normal n to the jump line must be the bisector of the angles $\chi_\pm$ between the vectors $\mathbf{j}$ and grad ω_2 on both sides of the jump line. The angles $\chi_\pm$ are given by the equations

$$\chi_+ = \arccos p, \quad \chi_- = \pi - \arccos p, \quad p = (\rho_{max} - \rho_{min})/(\rho_{max} + \rho_{min})$$

It can be shown directly that under these conditions both róots of Eq. (7.1) determine the same pair of principal axes. This agrees with the fact that Eq. (7.1) is invariant with respect to the substitution $\mathrm{tg}\,\gamma \to \mathrm{ctg}\,\gamma$, $\rho_1 \to \rho_2$, $\rho_2 \to \rho_1$. From now on we shall take the root $(\mathrm{tg}\,\gamma)_1$ without further qualification.

Let us suppose that $\rho_{max} \to \infty$ for a fixed ρ_{min}. In this case the root of Eq. (7.1) tends to the following limiting values (*) :

$$(\mathrm{tg}\,\gamma)_0 = \begin{cases} (\omega_{2y}/\omega_{2x})_0, & \text{if} \quad (\mathbf{j}\cdot\mathrm{grad}\,\omega_2)_0 > 0 \\ -(\zeta^1/\zeta^2)_0, & \text{if} \quad (\mathbf{j}\cdot\mathrm{grad}\,\omega_2)_0 < 0 \end{cases} \tag{7.2}$$

This result implies that the principal axes in the limiting case are oriented in such a way that either the vector grad ω_2 is parallel to the principal direction associated with the principal value ρ_{max}, or the vector $\mathbf{j}$ is parallel to the principal direction ρ_{min}. It is natural to expect that in the first of these cases $\mathbf{j}\cdot\mathrm{grad}\,\omega_2 = 0$ in the limiting case, since the vector $\mathbf{j}$ cannot have a component in the principal direction corresponding to an infinitely high resistivity (this statement will be proved below). We must therefore take the equation $\mathbf{j}\cdot\mathrm{grad}\,\omega_2 = 0$ instead of the first inequality of (7.2).

The second possibility has to do with the fact that the scalar froduct $\mathbf{j}\cdot\mathrm{grad}\,\omega_2$ is generally different from zero and negative. This is easy to see by recalling the interpretation of grad ω_2 as the fictitious electric field intensity vector in the problem for adjoint variables (see Sect. 2). Whereas the vector $\mathbf{j}$ (the current density) in the limiting case has a component along the principal direction ρ_{min} only, the vector grad ω_2 (the electric field intensity) generally has components in both principal directions. This means that the inequality sign applies in the second expression of (7.2) in the general case.

8. The asymptotic form of the optimal mode equations. The distinctive feature of these equations consists in the fact that they are of different form in those parts of the basic domain where the signs of the scalar product $\mathbf{j}\cdot\mathrm{grad}\,\omega_2$ are different. This conclusion can be readily arrived at on the basis of certain preliminary physical considerations and formulas (7.2).

Let us consider the matter in more detail.

Let us suppose that the resistivity of the medium is constant and isotropic ($\mathrm{P}_0 = \rho\mathrm{I}$, $\rho = \mathrm{const}$, I is an identity tensor). The vector lines $\mathbf{j}$ and grad ω_2 for this case are shown in Figs. 5a and 5b, respectively (Fig. 5c shows the corresponding graph of the magnitude $B\ (x)$ of the magnetic induction vector; we assume from now on that the function $B\ (x)$ is defined by a graph of this type).

Limiting ourselves once again to an isotropic (but inhomogeneous) resistivity distribution, we assume that the resistivity in some domain (e.g. the domain $CECC'E'C'$ in Figs. 5a and 5b) is $\rho_{min} = \mathrm{const}$, and that the resistivity in the remainder of the domain (to the left of $C'E'C'$ and to the right of CEC) is $\rho_{max} = \mathrm{const}$. Assuming that the topological structure of the lines $\mathbf{j}$ and grad ω_2 remains unchanged (except for the refraction of the lines at the boundary separating the domains), we find that this distribu-

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*) We are taking the limit with respect to the parameter $\mu = \rho_{min}/\rho_{max}$ which occurs explicitly in Eq. (7.1); the quantities ζ^1, ζ^2, ω_{2x}, ω_{2y} occurring in formula (7.2) depend on μ. This fact is reflected in the symbol $(\quad)_0$ used to denote scalar products.

tion to some extent agrees with the theorem of [1] characterizing the optimal disposition of the vector lines in the problem with an isotropic control.

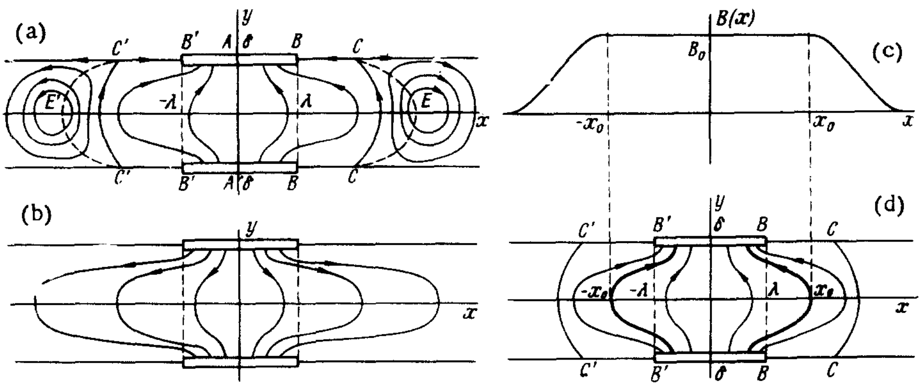

Fig. 5

In fact, the scalar product $\mathbf{j}\cdot\mathrm{grad}\ \omega_2$ is negative in the middle zone and positive in the side zones. The conditions of the theorem of [1] are then violated in the neighborhood of the lines CEC and $C'E'C'$ which serve as the control discontinuity lines in the problem with an isotropic control. It is also necessary for the Weierstrass condition to be fulfilled arbitrarily close to these lines , which means that this condition must be made compatible with the Weierstrass-Erdmann condition fulfilled along the lines of separation themselves. The conditions necessary for compatibility are set out in the footnote of Sect. 6. It is readily apparent that they cannot be fulfilled in the present case.

In fact, in the asymptotic case $\mu \to 1$ it is easy to point out limiting positions (which vanish for $\mu \doteq 1$) of the discontinuity lines CEC, $C'E'C'$. To do this we need merely superimpose Fig. 5a on Fig. 5b (both corresponding to the case $\rho = \mathrm{const}$) and connect with a smooth curve the points at which $\mathbf{j}\cdot\mathrm{grad}\ \omega_2 = 0$; this will be the required curve. It is also a simple matter to find the analytical equation of this curve. If the above conditions could be satisfied, then they would be satisfied for $\mu = 1$ by the exact solution corresponding to this case (e.g. see [4]). A direct check shows that this cannot happen, however.

The above fact obliges us to renounce attempts at finding the optimal solution in the class of isotropic solutions and requires us to include anisotropic controls P in our discussion (*).

This broader class of controls is remarkable in the fact that it enables us to alter the configuration of the vector lines $\mathbf{j}$ and $\mathrm{grad}\ \omega_2$ by altering the angle γ characterizing the orientation of the principal axes of the symmetric tensor P_0. We can assume here that the optimal solution is devoid of any control discontinuities; continuous variation of γ can yield the required configuration of vector lines, while the appearance of dis-

*) We spoke of positions of the lines separating the domains with differing values of the control as being asymptotic as $\mu \to 1$. This statement is not quite accurate; we cannot speak of asymptotic positions of nonexistent lines of separation. Our precise meaning will become clear below.

continuities entails certain rigid conditions which can be satisfied with great difficulty only (see Sect. 6).

Let us consider the asymptotic form of our basic differential equations.

The processes in the channel are described by system (1.4), or, which is the same thing, by the system

$$z_x^1 = \rho_{xx} z_y^2 - \rho_{xy} z_x^2, \qquad z_y^1 = \rho_{yx} z_y^2 - \rho_{yy} z_x^2 + c^{-1} V B \tag{8.1}$$

to which we must add the equations for the adjoint variables

$$\rho_1 \rho_2 \omega_{1x} = \rho_{xy} \omega_{2x} - \rho_{xx} \omega_{2y}, \qquad \rho_1 \rho_2 \omega_{1y} = \rho_{yy} \omega_{2x} - \rho_{yx} \omega_{2y} \tag{8.2}$$

The Cartesian components $\rho_{xx}, \rho_{xy} = \rho_{yx}, \rho_{yy}$ of the tensor P_0 are related to the principal values ρ_1, ρ_2 of this tensor and to the angle γ by formulas (1.1).

In seeking the asymptotic solution we shall assume that the quantities $z_x^1, \ldots, \omega_{2y}$ tend to finite nonzero limiting values as $\rho_{\max} \to \infty$. Let us set

$$\operatorname{tg} \gamma = (\operatorname{tg} \gamma)_0 + \mu m + O(\mu^2) \tag{8.3}$$

This expression is the expansion of the root of Eq. (7.1) in powers of the parameter μ. We note that the quantities $\zeta^1, \ldots, \omega_{2y}$ occurring in this equation must be given their exact values corresponding to a sufficiently large finite value of $\rho_{\max}$; expansion (8.3) must be effected in the parameter μ occurring explicitly in the coefficients of Eq. (7.1). Hence, for example (see the footnote in Sect. 7)

$$(\operatorname{tg} \gamma)_0 \neq \operatorname{tg} \gamma_0, \text{ but } (\operatorname{tg} \gamma)_0 \to \operatorname{tg} \gamma_0 \quad \text{as} \quad \mu \to 0$$

We obtain the following expansions for $\cos 2\gamma$, $\sin 2\gamma$:

$$\cos 2\gamma = (\cos 2\gamma)_0 - m (\sin 2\gamma)_0 [1 + (\cos 2\gamma)_0] \mu + O(\mu^2) \tag{8.4}$$

$$\sin 2\gamma = (\sin 2\gamma)_0 + m (\cos 2\gamma)_0 [1 + (\cos 2\gamma)_0] \mu + O(\mu^2) \tag{8.5}$$

The need to retain terms of order μ in these formulas is dictated by the structure of Eqs. (8.1), (8.2). In this connection we must bear in mind the fact that, according to the above remark, the quantities $(\cos 2\gamma)_0$, $(\sin 2\gamma)_0$ can also be expanded in series of the form

$$(\cos 2\gamma)_0 = \cos 2\gamma_0 + O(\mu), \qquad (\sin 2\gamma)_0 = \sin 2\gamma_0 + O(\mu)$$

In computing the values of $(\operatorname{tg} \gamma)_0$ and m we must use the first formula of (7.2) for those points where $(\mathbf{j} \cdot \operatorname{grad} \omega_2)_0 > 0$ and the second formula for those points where $(\mathbf{j} \cdot \operatorname{grad} \omega_2)_0 < 0$. After some simple operations we obtain

$$(\operatorname{tg} \gamma)_0 = \frac{z_y^2}{z_x^2}, \qquad m = - \frac{(\operatorname{grad} z^2)^2 (\operatorname{grad} \omega_2 \cdot \operatorname{grad} z^2)}{(\omega_{2y} z_x^2 - \omega_{2x} z_y^2)(z_x^2)^2} \tag{8.6}$$

for the domain where $(\mathbf{j} \cdot \operatorname{grad} \omega_2)_0 < 0$ and

$$(\operatorname{tg} \gamma)_0 = \frac{\omega_{2y}}{\omega_{2x}}, \qquad m = \frac{(\operatorname{grad} \omega_2)^2 (\operatorname{grad} \omega_2 \cdot \operatorname{grad} z^2)}{(\omega_{2y} z_x^2 - \omega_{2x} z_y^2)(\omega_{2x})^2} \tag{8.7}$$

for the domain where $(\mathbf{j} \cdot \operatorname{grad} \omega_2)_0 > 0$.

It is now an easy matter to write out the required asymptotic equations. Let us consider the case $(\mathbf{j} \cdot \operatorname{grad} \omega_2)_0 < 0$. Substituting expansions (8.4), (8.5) into Eq. (8.1) and making use of formulas (8.6), we find that the coefficient of $\rho_1 = \rho_{\max}$ in the right side of Eqs. (8.1) vanishes identically. We must pass to the limiting case $\mu = 0$ in the remaining terms. The expressions for $z_x^1, \ldots, \omega_{2y}$ then become the correponding limiting quantities for which we retain the previous symbols $z_x^1, \ldots, \omega_{2y}$. As regards Eqs. (8.2), the coefficient of $\rho_1 = \rho_{\max}$ in the right side of each of them is not equal to zero; dividing both sides of each of these equations by ρ_{max}, we pass to the limiting case $\mu = 0$. This gives us the equations describing the optimal process in the domain $\mathbf{j}\cdot\mathrm{grad}\ \omega_2 < 0$,

$$z_x^1 = \rho_{min} z_y^2 + \rho_{min} z_x^2 K, \qquad z_y^1 = -\rho_{min} z_x^2 + \rho_{min} z_y^2 K + c^{-1}VB \tag{8.8}$$

$$\rho_{min}\omega_{1x} = -z_x^2 \frac{\omega_{2y} z_x^2 - \omega_{2x} z_y^2}{(\mathrm{grad}\ z^2)^2}, \qquad \rho_{min}\omega_{1y} = -z_y^2 \frac{\omega_{2y} z_x^2 - \omega_{2x} z_y^2}{(\mathrm{grad}\ z^2)^2} \tag{8.9}$$

In the domain where $\mathbf{j}\cdot\mathrm{grad}\ \omega_2 > 0$ we have the equations (*)

$$z_x^1 = -\frac{\omega_{2y} z_x^2 - \omega_{2x} z_y^2}{(\mathrm{grad}\ \omega_2)^2}(\rho_{max}\omega_{2x} - 2\rho_{min}\omega_{2y}K + \rho_{min}\omega_{2x}K^2) \tag{8.10}$$

$$z_y^1 = -\frac{\omega_{2y} z_x^2 - \omega_{2x} z_y^2}{(\mathrm{grad}\ \omega_2)^2}(\rho_{max}\omega_{2y} - 2\rho_{min}\omega_{2x}K + \rho_{min}\omega_{2y}K^2) + c^{-1}VB$$

$$\rho_{max}\omega_{1x} = -\omega_{2y} + K\omega_{2x}, \qquad \rho_{max}\omega_{1y} = \omega_{2x} + K\omega_{2y} \tag{8.11}$$

In these formulas

$$K = \frac{\mathrm{grad}\ \omega_2\cdot\mathrm{grad}\ z^2}{\omega_{2y} z_x^2 - \omega_{2x} z_y^2} \tag{8.12}$$

9. The asymptotic solution in the domain $\mathbf{j}\cdot\mathrm{grad}\ \omega_2 < 0$. Let us consider Eqs. (8.8) and (8.9) in more detail. The latter pair of equations implies that

$$z^2 = h(\omega_1) \tag{9.1}$$

where $h(\omega_1)$ is an arbitrary function.

Equations (8.9) are equivalent to the system

$$\omega_{2x} = \rho_{min}(\omega_{1y} - \omega_{1x}K), \qquad \omega_{2y} = -\rho_{min}(\omega_{1x} + \omega_{1y}K) \tag{9.2}$$

The set of equations (8.8), (8.12) and (9.2) is obtainable in a simpler way. Let us assume from the very beginning that $\rho_1 = \rho_{max} = \infty$, $\rho_2 = \rho_{min}$. The basic equations of the problem can then be written as

$$0 = z_y^2\cos\gamma - z_x^2\sin\gamma \tag{9.3}$$
$$z_y^1\cos\gamma - z_x^1\sin\gamma = -\rho_{min}(z_y^2\sin\gamma + z_x^2\cos\gamma) + c^{-1}VB\cos\gamma$$

The first of these equations expresses the equality to zero of the component of the vector $\mathbf{j}$ along the α-axis; the second equation expresses the differential Ohm law along the β-axis.

An equivalent expression of system (9.3) can be obtained by complementing the equation

$$\mathrm{tg}\,\gamma = z_y^2 / z_x^2 \tag{9.4}$$

with the system

$$z_x^1 = \rho_{min} z_y^2 + \rho_{min} z_x^2 K, \qquad z_y^1 = -\rho_{min} z_x^2 + \rho_{min} z_y^2 K + c^{-1}VB \tag{9.5}$$

where K represents an arbitrary function interpreted as a control.

The next step is to pose the initial optimal problem for system (9.5). The Euler equations then become

$$\omega_{2x} = \rho_{min}(\omega_{1y} - \omega_{1x}K), \qquad \omega_{2y} = -\rho_{min}(\omega_{1x} + \omega_{1y}K) \tag{9.6}$$

The steadystate condition with respect to the control K gives us the equation

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*) Equations (8.10), (8.11) imply that $\mathbf{j}\cdot\mathrm{grad}\ \omega_2 \to 0$ as $\rho_{max}\to\infty$.

$$z^2 = h(\omega_1) \tag{9.7}$$

It is clear that Eqs. (9.5)–(9.7) are equivalent to Eqs. (8.8), (8.12), (9.2).

Let us rewrite the basic equations in a form more convenient for future use. To this end we eliminate the function z^2 from system (8.8) with the aid of (9.1) to obtain

$$z^1_x = \rho_{\min} h'(\omega_1)(\omega_{1y} + \omega_{1x}K), \qquad z^1_y = \rho_{\min} h'(\omega_1)(\omega_{1y}K - \omega_{1x}) + c^{-1} VB \tag{9.8}$$

Eliminating the function z^1 from this system, we obtain the equation

$$h'(\omega_1)[\Delta\omega_1 + \omega_{1x}K_y - \omega_{1y}K_x] + h''(\omega_1)(\operatorname{grad}\omega_1)^2 = (c\rho_{\min})^{-1} VB_x(x) \tag{9.9}$$

On the other hand, system (9.2) generates the equation

$$\Delta\omega_1 = \omega_{1x}K_y - \omega_{1y}K_x \tag{9.10}$$

This equation together with (9.9) yields the relation

$$2h'(\omega_1)\ \Delta\omega_1 + h''\ (\omega_1)\ (\operatorname{grad}\ \omega_1)^2 = (c\rho_{\min})^{-1}\ VB_x\ (x) \tag{9.11}$$

If the function $h(\omega_1)$ is known, then Eqs. (9.10) and (9.11) together with the corresponding boundary conditions determine the functions ω_1 and K.

Let us consider the boundary conditions at the electrodes. The necessary conditions

$$z^1 = z^1_{\pm} = \text{const}, \qquad \omega_2 = \omega_{2\pm} = \text{const} \qquad \text{for } |x| < \lambda,\ y = \pm\delta$$

imply the formulas (see (9.2) and (9.8))

$$h'(\omega_1)(\omega_{1y} + \omega_{1x}K) = 0, \qquad \omega_{1y} - \omega_{1x}K = 0$$

which are valid along the electrodes. Disregarding the trivial possibility that $h'(\omega_1) = 0$ (which means that $j_y = 0$ at the electrodes), we infer from this that the following equations are valid along the electrodes:

$$\omega_{1y} = 0, \qquad K = 0 \tag{9.12}$$

Thus, the current lines $\mathbf{j}$ (and along with them the principal directions β) must be normal to the electrodes. Symmetry considerations indicate that relations (9.12) are fulfilled along the x-axis as well.

The insulating walls of the channel are the current lines $\mathbf{j}$; this fact is expressed mathematically by the equations

$$\omega_1(x, \pm\delta)|_{x<-\lambda} = \omega_{1-} = \text{const}, \qquad \omega_1(x, \pm\delta)|_{x>\lambda} = \omega_{1+} = \text{const} \tag{9.13}$$

The symmetry of the problem indicates that the segment AA (Fig. 5a) is also a current line; we have

$$\omega_1 = {}^1/_2(\omega_{1+} + \omega_{1-}) \ \ (\text{along} \quad AA) \tag{9.14}$$

It remains for us to write out the equations relating the values of the functions ω_1 and $h(\omega_1)$ with the parameters of the external circuit. These conditions are of the form (see formulas (1.6) and (2.13))

$$z^1_+ - z^1_- = R(z^2_+ - z^2_-), \qquad \omega_{2+} - \omega_{2-} + 1 = R(\omega_{1+} - \omega_{1-}) \tag{9.15}$$

We can transform these two equations by means of formulas (9.2), (9.8) and (9.14). Let us consider the current line L ($\omega_1 = \text{const}$) connecting the electrodes and compute the line integral

$$\int_L \omega_{2x}\, dx + \omega_{2y}\, dy$$

along this current line. Making use of Eqs. (9.2), we obtain

$$\int_L \omega_{2x}\, dx + \omega_{2y}\, dy = \rho_{\min} \int_L \omega_{1y}\, dx - \omega_{1x}\, dy$$

Substituting this result into the second equation of (9.15), we find that

$$\rho_{\min} \int\limits_L \omega_{1y}\, dx - \omega_{1x}\, dy = R\,(\omega_{1+} - \omega_{1-}) - 1 \qquad (9.16)$$

In exactly the same way we can use Eqs. (9.8) to obtain the formula

$$\rho_{\min} h'(\omega_1) \int\limits_L \omega_{1y}\, dx - \omega_{1x}\, dy = R\,[h(\omega_{1+}) - h(\omega_{1-})] - c^{-1} \int\limits_L VB\, dy \quad (9.17)$$

Combining (9.16) and (9.17), we obtain the formula

$$h'(\omega_1) = \frac{1}{R\,(\omega_{1+} - \omega_{1-}) - 1} \left\{ R\,[h(\omega_{1+}) - h(\omega_{1-})] - c^{-1} \int\limits_L VB\, dy \right\} \quad (9.18)$$

$$(\omega_{1-} \leqslant \omega_1 \leqslant \omega_{1+})$$

which defines the function $h(\omega_1)$ implicitly. Complementing condition (9.18) with Eq. (9.16) taken from the parameter value $\omega_1 = (^1/_2)\,(\omega_{1+} + \omega_{1-})$ and adding the remaining boundary conditions to this combination, we obtain the complete system of relations which together with (9.11) and (9.10) determine the functions ω_1 and K. We assume here that the function ω_1 cannot take on values outside the interval $[\omega_{1-}, \omega_{1+}]$. If this condition is not fulfilled, the picture is made more complicated by the appearance of closed current lines.

Note. If condition (9.18) is fulfilled, then it is sufficient to take Eq. (9.16) for a single value (e.g. $\omega_1 = {}^1/_2\,(\omega_{1+} + \omega_{1-})$) of the parameter ω_1, since its validity for other values follows automatically.

To prove this let us integrate both sides of Eq. (9.11) over the domain Σ bounded by the two current lines

$$L_0\,(\omega_1 = (^1/_2)\,(\omega_{1+} + \omega_{1-})), \qquad L_1\,(\omega_1 = \omega_1^\circ), \qquad (^1/_2)\,(\omega_{1+} + \omega_{1-}) < \omega_1^\circ \leqslant \omega_{1+}$$

and by the two electrode segments (Fig. 6). Making use of Eq. (9.18) and condition (9.16) taken for the parameter value $\omega_1 = (^1/_2)\,(\omega_{1+} + \omega_{1-})$, we can write the result as

$$\iint\limits_\Sigma [h'(\omega_1^\circ) + h'(\omega_1)]\,\Delta\omega_1 dx dy = 0$$

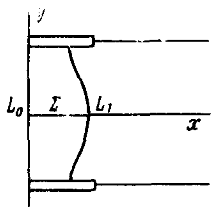

Fig. 6

This equation is valid for any domain Σ of the indicated type, which means that the integrand in the above expression can be written as $\frac{\partial}{\partial y}(u\omega_{1x}) - \frac{\partial}{\partial x}(u\omega_{1y})$ (where the function u vanishes at the electrodes). From this we can readily infer the required result.

The initial requirement that $(\mathbf{j}\cdot\mathrm{grad}\ \omega_2)_0 < 0$ becomes the inequality $\mathbf{j}\cdot\mathrm{grad}\ \omega_2 < 0$ in the limiting case; formulas (9.1) and (9.2) imply the equivalent condition

$$h'(\omega_1) > 0, \qquad \forall \omega_1 \in (\omega_{1-}, \omega_{1+}) \qquad (9.19)$$

If we also assume that the optimal solution is such that inequality (9.19) is fulfilled throughout the domain through which the currents flow, then, summarizing our results, we arrive at the following problem.

We are required to find the function $\omega_1(x, y)$ which satisfies Eq. (9.11) under boundary conditions (9.12), (9.13), the condition

$$-\rho_{\min} \int\limits_{-\delta}^{\delta} \omega_{1x}(0, y)\, dy = R\,(\omega_{1+} - \omega_{1-}) - 1 \qquad (9.20)$$

and additional requirements (9.18), which together with inequality (9.19) determines the function $h(\omega_1)$. The integral in the right side of Eq. (9.19) must be taken along the current line L corresponding to that value of the parameter $\omega_1 = \text{const}$ which serves as the argument of the function $h'(\omega_1)$ on the left side of this equation. The current lines $\omega_1 = \omega_{1+}$, $\omega_1 = \omega_{1-}$ are critical; the function $h'(\omega_1)$ equals zero along these lines.

Once the function $\omega_1(x, y)$ has been determined, Eq. (9.10) together with the Cauchy condition $K(x, \pm\delta)_{|x|<\lambda} = 0$ (see (9.12)) determines the function K.

If the function $B(x)$ retains a constant value B_0 over some range of values of the argument, then the problem can be simplified somewhat.

Let us consider the current lines L lying entirely in the domain where the condition $B(x) = B_0 = \text{const}$ is fulfilled. Formula (9.18) indicates that the function $h'(\omega_1)$ assumes one and the same constant value on such lines; Eq. (9.11) reduces to the Laplace equation in the corresponding domain. But Eq. (9.10) then has the integral

$$K = f(\omega_1) \tag{9.21}$$

where $f(\omega_1)$ is an arbitrary function. Since the domain $B(x) = B_0 = \text{const}$ includes the electrodes (*) (Fig. 5c) along which $K = 0$, it follows that we must set $f \equiv 0$, i.e. $K \equiv 0$, in formula (9.21).

This result implies that the optimal control is isotropic and equal to $\rho_{\min} = \text{const}$ in the domain occupied by the current lines lying entirely within the zone $B(x) = B_0 = \text{const}$. Anisotropic controls are characterized by nonzero values of the function K; the control is anisotropic in the domains containing current lines some part of which lies in the zone $B(x) \neq \text{const}$. The thick curve in Fig. 5d is the current line which separates the isotropic and anisotropic control zones. The current lines $BCCB$ and $B'C'C'B'$ are critical.

Our supposition of the existence of critical current lines can be substantiated by analyzing formula (9.18). The first factor appearing in the right side of this equation is always negative (it is the inverse difference between the voltage drop between the electrodes in the problem for adjoint variables and the external unit electromotive force which produces this voltage drop). As regards the second factor, its sign generally depends on the choice of the current line L. Let the function $B(x)$ be defined by the curve in Fig. 5c, and let $V = V(y)$. If the current line lies wholly within the zone where $B(x) = B_0 = \text{const}$, then the expression in the numerator is negative (it is clear that the term $R \; [h(\omega_{1+}) - h(\omega_{1-})]$ does not exceed the expression

$$\frac{B_0}{c}\int_{-\delta}^{\delta} V\,dy$$

if the function $B(x)$ is defined as in Fig. 5c, since this term does not exceed the above expression even if $B(x) \equiv B_0 = \text{const}$ everywhere in the channel). On the other hand, if the current line (connecting the electrodes in accordance with our hypothesis) lies in large measure in the domain $B(x) = 0$, then the numerator of ratio (9.21) is positive. The latter variant must be discarded as one which contradicts inequality (9.19).

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*) This statement follows from the above supposition whereby there exist current lines which connect the electrodes and lie entirely in the domain $B(x) = B_0 = \text{const}$.

This implies that the current lines in the adopted arrangement do not penetrate far enough into the zone of decay of the field $B(x)$; moreover, it is possible for the entire domain occupied by the currents to be bounded by the critical lines $BCCB$ and $B'C'C'B'$ (along which $h'(\omega_1) = 0$) and by the electrodes. Outside this domain either the topological structure of the current lines changes (this change being manifested in the appearance of closed lines) or the process is described by Eqs. (8.10), (8.11) corresponding to the case $\mathbf{j}\cdot\mathrm{grad}\,\omega_2 > 0$. It is remarkable that none of these possibilities has any effect on the final result, since the condition $h'(\omega_1) = 0$ at the critical current line is sufficient to determine this line regardless of what happens on the other side of it. Nothing prevents us from assuming, for example, that the channel outside the domain $CCC'C'$ is filled with a homogeneous isotropic insulator $\rho = \infty$, so that there are no currents at all outside the domain $CCC'C'$.

The coefficient of the principal part in Eq. (9.11) vanishes at the critical current lines. The control is always anisotropic sufficiently close to these lines, since the critical lines necessarily extend beyond the zone $B(x) = B_0 = \mathrm{const}$. Such small neighborhoods of the critical lines are not, however, characterized by rapid variations of the function ω_1. In fact, the derivatives ω_{1x}, ω_{1y} assume finite values along the critical lines (this follows either from formula (9.16) or from formula (9.17) with allowance for the fact that the factor $h'(\omega_1)$ and the expression on the right side vanish at the critical lines).

Appendix. Concerning the derivation of inequality (3.6). Let us transform the expression in square brackets in the left side of inequality (3.5). The components J_α, J_β of the vector $\mathbf{J}$ can be expressed in terms of the components J_n, J_t by means of the formulas (Fig. 2)

$$J_\alpha = J_n\cos\theta - J_t\sin\theta, \qquad J_\beta = J_n\sin\theta + J_t\cos\theta \tag{A.1}$$

Replacing J_n, J_t by their expressions (3.3), we obtain

$$J_\alpha = j_\alpha - \frac{\rho_{tt} - P_{tt}}{P_{tt}} j_t\sin\theta - \frac{\rho_{tn} - P_{tn}}{P_{tt}} j_n\sin\theta + O(\varepsilon) \tag{A.2}$$

$$J_\beta = j_\beta + \frac{\rho_{tt} - P_{tt}}{P_{tt}} j_t\cos\theta + \frac{\rho_{tn} - P_{tn}}{P_{tt}} j_n\cos\theta + O(\varepsilon)$$

The following formulas are valid (see (1.1) and Fig. 2):

$$\rho_{tt} = {}^1\!/_2\,[\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\theta], \quad P_{tt} = {}^1\!/_2\,[P_1 + P_2 - (P_1 - P_2)\cos 2\psi]$$

$$\rho_{nn} = {}^1\!/_2\,[\rho_1 + \rho_2 + (\rho_1 - \rho_2)\cos 2\theta], \quad P_{nn} = {}^1\!/_2\,[P_1 + P_2 + (P_1 - P_2)\cos 2\psi] \tag{A.3}$$

$$\rho_{tn} = \rho_{nt} = -{}^1\!/_2\,(\rho_1 - \rho_2)\sin 2\theta, \quad P_{tn} = P_{nt} = -{}^1\!/_2\,(P_1 - P_2)\sin 2\psi$$

We can use these relations to express the right sides of Eqs. (A.2) in terms of the principal values ρ_1, ρ_2, P_1, P_2 of the tensors P_0 and P and in terms of the angles θ and ψ between the normal $\mathbf{n}$ to the strip of variation and the axes α and A, respectively. We obtain

$$J_\alpha = [P_1 + P_2 - (P_1 - P_2)\cos 2\psi]^{-1}\{m - 2(P_1 - P_2)\, j_n \sin\psi\sin(\theta - \psi)\} \tag{A.4}$$
$$J_\beta = [P_1 + P_2 - (P_1 - P_2)\cos 2\psi]^{-1}\{n - 2(P_1 - P_2)\, j_n \cos\psi\sin(\theta - \psi)\}$$

(the terms $O(\varepsilon)$ are omitted from these and subsequent expressions).

In these formulas ($\delta\rho_i = P_i - \rho_i$, $i = 1, 2$)

$$m = j_\alpha\,[P_1 + P_2 - (P_1 - P_2)\cos 2\theta] - 2\sin\theta\,[\delta\rho_1 j_\alpha \sin\theta - \delta\rho_2 j_\beta \cos\theta] \qquad (A.5)$$
$$n = j_\beta\,[P_1 + P_2 - (P_1 - P_2)\cos 2\theta] + 2\cos\theta\,[\delta\rho_1 j_\alpha \sin\theta - \delta\rho_2 j_\beta \cos\theta]$$

Inequality (3.5) contains the components $P_{\alpha\alpha}$, $P_{\alpha\beta} = P_{\beta\alpha}$, $P_{\beta\beta}$ of the tensor P in the system of principal axes α, β of the tensor P_0. These components can be expressed in terms of the principal values of P_1, P_2 of the tensor P and in terms of the angle $\lambda = \theta - \psi$ between the principal axes α and A of the tensors P_0 and P by means of formulas (1.1) with the axis x replaced by the axis α, and the axis α by the axis A . Here $\gamma = \lambda$. After some simple operations we obtain

$$(P_{\alpha\alpha} - \rho_1) J_\alpha + P_{\alpha\beta} J_\beta = (P_1 - \rho_1) J_\alpha - (P_1 - P_2)\sin\lambda\,(J_\alpha \sin\lambda - J_\beta \cos\lambda) \qquad (A.6)$$
$$P_{\beta\alpha} J_\alpha + (P_{\beta\beta} - \rho_2) J_\beta = (P_2 - \rho_2) J_\beta + (P_1 - P_2)\sin\lambda\,(J_\alpha \cos\lambda + J_\beta \sin\lambda)$$

Eliminating J_α, J_β with the aid of (A.4), we obtain the formulas

$$[(P_{\alpha\alpha} - \rho_1) J_\alpha + P_{\alpha\beta} J_\beta]\,[P_1 + P_2 - (P_1 - P_2)\cos 2\psi] = (P_1 - \rho_1)\, m - (P_1 - P_2)\, f \sin\lambda \qquad (A.7)$$
$$[P_{\beta\alpha} J_\alpha + (P_{\beta\beta} - \rho_2) J_\beta]\,[P_1 + P_2 - (P_1 - P_2)\cos 2\psi] = (P_2 - \rho_2)\, n + (P_1 - P_2)\, g \sin\lambda$$

Here

$$f = [2\rho_1 j_\alpha + (\rho_1 - \rho_2)\, j_\beta \sin 2\theta]\sin\lambda - j_\beta\,[\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\theta]\cos\lambda \qquad (A.8)$$
$$g = [2\rho_2 j_\beta + (\rho_2 - \rho_1)\, j_\alpha \sin 2\theta]\sin\lambda + j_\alpha\,[\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\theta]\cos\lambda$$

Constructing the left side of inequality (3.5) by means of the above formulas and recalling steadystate condition (2.9), we arrive at inequality (3.6).

Concerning the derivation of formula (6.9) etc. The limiting values of the vectors j and grad ω_2 on the two sides of the line of discontinuity of the controls are related by expressions obtainable from initial equations (1.1) and the continuity conditions

$$[z_t^1]_-^+ = [z_t^2]_-^+ = [\omega_{1t}]_-^+ = [\omega_{2t}]_-^+ = 0 \qquad (A.9)$$

Let $\theta_\pm$ be the angles between the normal $\mathbf{n}$ to the jump line and the axes $\alpha_\pm$ of the tensors $(P_0)_\pm$ on the opposite sides of this line. To obtain these relations we must write out Eqs. (1.1) in the principal axes of the tensor P_0. We have

$$z_\alpha^1 = -\rho_1 j_\alpha + c^{-1} VB \sin\gamma, \qquad z_\beta^1 = -\rho_2 j_\beta + c^{-1} VB \cos\gamma \qquad (A.10)$$

Here $j_\alpha = -z_\beta^2$, $j_\beta = z_\alpha^2$ and γ is the angle between the axis α and the axis x (Fig. 2).

Next, inverting formulas (A.1) and applying Eqs. (2.6), (2.7) for the values of $(\omega_{1t})_\pm$ and Eqs. (A.10) for the values of $(z_t^1)_\pm$, we can write Eqs. (A.9) in the form

$$[\rho_1 j_\alpha \sin\theta]_-^+ - [\rho_2 j_\beta \cos\theta]_-^+ = 0, \qquad [\rho_1^{-1}\omega_{2\alpha}\cos\theta]_-^+ + [\rho_2^{-1}\omega_{2\beta}\sin\theta]_-^+ = 0 \qquad (A.11)$$
$$[j_\alpha \cos\theta]_-^+ + [j_\beta \sin\theta]_-^+ = 0, \qquad [\omega_{2\alpha}\sin\theta]_-^+ - [\omega_{2\beta}\cos\theta]_-^+ = 0$$

The required relations can be obtained by solving these equations for $(j_\alpha)_-$, $(j_\beta)_-$, $(\omega_{2\alpha})_-$, $(\omega_{2\beta})_-$.

Concerning the derivation of formulas (7.1). The root $(\mathrm{tg}\,\gamma)_1$ of Eq. (7.1) is given by the formula

$$\frac{(\rho_1 + \rho_2)(\omega_{2y}\zeta^2 - \omega_{2x}\zeta^1)}{2(\rho_1\zeta^2\omega_{2x} + \rho_2\zeta^1\omega_{2x})} + \qquad (A.12)$$
$$+ \frac{\{(\rho_1 + \rho_2)^2(\omega_{2y}\zeta^2 - \omega_{2x}\zeta^1)^2 + 4(\rho_1^2 + \rho_2^2)\,\omega_{2x}\omega_{2y}\zeta^1\zeta^2 + 4\rho_1\rho_2[(\omega_{2y}\zeta^1)^2 + (\omega_{2x}\zeta^2)^2]\}^{1/2}}{2(\rho_1\zeta^2\omega_{2x} + \rho_2\zeta^1\omega_{2y})}$$

or, which is the same thing, by the expression (see Fig. 7)

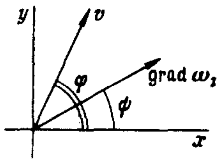

Fig. 7

$$\frac{-(\rho_1+\rho_2)\cos(\varphi+\psi)+[4\rho_1\rho_2+(\rho_1-\rho_2)^2\cos^2(\varphi-\psi)]^{1/2}}{2(\rho_1\sin\varphi\cos\psi+\rho_2\cos\varphi\sin\psi)} \qquad (A.13)$$

It is easy to see that the expression

$$\frac{1}{(\cos^2\gamma)_1}4(\rho_1\sin\varphi\cos\psi+\rho_2\cos\varphi\sin\psi)^2(|j|\,\mathrm{grad}\,\omega_2|)^{-1}j_\alpha\,\omega_{2\alpha}$$

can be rewritten as

$$\alpha_1\rho_1^2+\alpha_2\rho_2^2+2\alpha_3\rho_1\rho_2+2(\alpha_4\rho_1+\alpha_5\rho_2)\;[4\rho_1\rho_2+(\rho_1-\rho_2)^2\cos^2(\varphi-\psi)]^{1/2} \qquad (A.14)$$

Simple operations give us the following expressions for the coefficients:

$$\alpha_1=2\sin^2\varphi\cos(\varphi-\psi),\quad \alpha_2=2\sin^2\psi\cos(\varphi-\psi),\qquad \alpha_4=\sin^2\varphi,\quad \alpha_5=\sin^2\psi$$
$$\alpha_3=4\sin\varphi\sin\psi-(\sin^2\varphi+\sin^2\psi)\cos(\varphi-\psi)$$

The sign of $j_\alpha\omega_{2\alpha}$ is the same as that of expression (A.13); the latter is always positive, since the difference

$$[4\rho_1\rho_2+(\rho_1-\rho_2)^2\cos^2(\varphi-\psi)]\,(\rho_1\sin^2\varphi+\rho_2\sin^2\psi)^2-\{(\rho_1^2\sin^2\varphi+\rho_2^2\sin^2\psi)\times$$
$$\times\cos(\varphi-\psi)+\rho_1\rho_2[4\sin\varphi\sin\psi-(\sin^2\varphi+\sin^2\psi)\cos(\varphi-\psi)]\}^2$$

is clearly equal to

$$4\rho_1\rho_2\,[\rho_1\sin^2\varphi-\rho_2\sin^2\psi-(\rho_1-\rho_2)\sin\varphi\sin\psi\cos(\varphi-\psi)]^2$$

The proof for the quantity $j_\beta\omega_{2\beta}$ is entirely analogous.

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